SOME CONSEQUENCES OF A COVARIANT DIFFERENTIAL COORDINATE SYSTEM

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Lieutenant, United States Naval Reserve

Submitted in partial fulfillment of

the requirements for the degree of

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ABSTRACT

The assumption is made that the concept of a covariant coordinate system is meaningful and that such a system must satisfy the differential equations

a) $$dx_i = g_{ij}\, dx^j$$

where the g_{ij} are the components of the metric tensor in the corresponding contravarient system.

Using this system it is shown that the derivatives of tensors by these new variables are just those quantities that would result from the raising of indices in the derivatives by the corresponding contravariant variables.

The requirement that the equations a) be solvable for a set of covariant variables x_i, namely

b) $$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j},$$

produces a radical change in the nature of the Christoffel symbols, making them triply symetric. The equations b) also permit a new physical interpretation of the symbols of the second kind as a direct dual of the symbols of the first kind,

c) $$[ij, k] = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} \text{ and}$$

d) $$\left\{ {k \atop ij} \right\} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x_k}.$$

This dualism is applied to the usual expression for the LaPlacean of a function and the resulting symetry noted.

Several approaches are made to apply the above results to the Riemann-Christoffel tensor. The expression for this tensor is simplified in form

by the use of c) and d). It is noted that the Riemann-Christoffel tensor is now a first order set of non-linear differential equations in the g_{ij}; however the solution appears to be just as difficult. In particular, the method of solution of the Einstein Field Equations by assuming a linear approximation for the g_{ij} can no longer be applied with any degree of simplicity.

The simplification of the form of the Riemann-Christoffel tensor does not alter the number of independent components of this tensor, nor does it change the nature of the various contractions of this tensor. These conclusions are reached by direct calculation of the twenty components and by the application of some elements of group representation theory following a technique used by D. E. Littlewood.

The author wishes to express his appreciation for the assistance and encouragement given him in this investigation by Professor Charles C. Torrance of the U. S. Naval Postgraduate School.

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I. Introduction

This paper is the result of an attempt to make the dualism in modern tensor theory a bit more complete. It is also an attempt to make some of the artificial definitions and strategems which are presently employed in this dualism less artificial. The starting point for this paper is to impose a dualism on one of the simplest of the tensors of rank one, the differential dx^i of the coordinate variable x^i.

Let us define the quantities x_i by the set of differential equations

1) $$dx_i = g_{ij}\, dx^j.$$

The requirement that the x_i be interpretable as coordinates is the requirement that the equations 1) be completely integrable, so that a unique solution exists. By applying the usual argument of calculus, this is equivalent to satisfying the set of equations

2) $$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j}$$

for all i, j, k. It is also trivially evident that 1) implies

3) $$\frac{\partial x_i}{\partial x^j} = g_{ij}\,.$$

If the x_i are to be coordinates, then the x_i are unique functions of the x^j such that

$$x_i = x_i(x^j).$$

If these are functions, then the inverse equations

4) $$x^i = x^i(x_j)$$

must exist almost everywhere, so that

5) $$dx^i = \frac{\partial x^i}{\partial x_j}\, dx_j.$$

But from 1)

$$g^{ij} dx_i = g^{ij} g_{ik} dx^k$$
$$= \delta^j_k dx^k .$$

Thus $$g^{ij} dx_i = dx^j .$$

Hence

6) $$\frac{\partial x^j}{\partial x_i} = g^{ij} .$$

Formulas 3) and 6) constitute an important dualism, and give a physical significance to g^{ij} which is lacking when only a contravariant coordinate system is assumed.

It is perhaps important at this point to note that a distinction must be made as to which coordinate system the g_{ij} and the g^{ij} refer. Since g^{ij} is a tensor of the second order, its representation in a new coordinate system under a change of coordinates is, by definition,

$$g^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} g^{ij} .$$

So let us apply this rule to calculation of the g^{ij}, given in the contravariant system, as it would appear in the covariant system.

7) $$g_{\alpha\beta} = \frac{\partial x_\alpha}{\partial x^i} \frac{\partial x_\beta}{\partial x^j} g^{ij} .$$

In other words, the tensor g^{ij} in the contravariant system would appear as $g_{\alpha\beta}$. Likewise one can show that the dual of g_{ij} in the contravariant system is $g^{\alpha\beta}$. So in future calculations, the coordinate system in which the g's are given will be the contravariant system unless otherwise noted.

II. The dualism in Section I has an effect on the derivatives of tensors, both with respect to the contravariant and the covariant coordinates. Harris [1] has shown that if ϕ is a tensor of order zero, then

8) $$\frac{\partial \phi}{\partial x_i} = g^{ij} \frac{\partial \phi}{\partial x^j},$$

and that if $V_i = g_{ij} V^j$, then

9) $$\frac{\partial V_\alpha}{\partial x_i} = g^{ij} \frac{\partial V_\alpha}{\partial x^j},$$

where the g^{ij} are the components of the metric tensor in terms of the contravariant variable, as discussed in Section I.

Using these it is our purpose to show in this section that there is a contravariant derivative, which is a tensor, and that this derivative is simply related to the covariant derivative. We will show that

10) $$T^{i,j} = \frac{\partial T^i}{\partial x_j} + T^k g^{jh} \left\{ {i \atop kh} \right\}$$

is the contravariant derivative. In the next section we show that this expression simplifies slightly by expressing the Christoffel symbols in terms of the covariant coordinates.

Let
$$T^\alpha = \frac{\partial x^\alpha}{\partial x^i} T^i$$

$$\frac{\partial T^\alpha}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}\left[\frac{\partial x^\alpha}{\partial x^i} T^i\right]$$

$$= \frac{\partial x^\alpha}{\partial x^i}\left[\frac{\partial}{\partial x_\beta} T^i\right] + T^i \frac{\partial}{\partial x_\beta}\left[\frac{\partial x^\alpha}{\partial x^i}\right]$$

$$= \frac{\partial x^\alpha}{\partial x^i}\left[\frac{\partial T^i}{\partial x^j} \frac{\partial x^j}{\partial x_\beta}\right] + T^i \frac{\partial x^j}{\partial x_\beta}\left[\frac{\partial^2 x^\alpha}{\partial x^i \partial x^j}\right]$$

$$= T^i \frac{\partial x^j}{\partial x_\beta}\left[\left\{{k \atop ij}\right\}\frac{\partial x^\alpha}{\partial x^k} - \left\{{\alpha \atop \gamma\delta}\right\}\frac{\partial x^\gamma}{\partial x^i}\,\frac{\partial x^\delta}{\partial x^j}\right]$$

$$+ \frac{\partial x^\alpha}{\partial x^i}\,\frac{\partial x^j}{\partial x_\beta}\,\frac{\partial T^i}{\partial x^j}$$

$$= T^i \frac{\partial x^j}{\partial x_\beta}\left\{{k \atop ij}\right\}\frac{\partial x^\alpha}{\partial x^k} + \frac{\partial x^\alpha}{\partial x^i}\,\frac{\partial x^j}{\partial x_\beta}\,\frac{\partial T^i}{\partial x^j}$$

$$- T^i \frac{\partial x^\gamma}{\partial x^i}\cdot\frac{\partial x^j}{\partial x_\beta}\,\frac{\partial x^\delta}{\partial x^j}\left\{{\alpha \atop \gamma\delta}\right\}$$

$$= \left[T^k\left\{{i \atop kj}\right\} + \frac{\partial T^i}{\partial x^j}\right]\frac{\partial x^\alpha}{\partial x^i}\cdot\frac{\partial x^j}{\partial x_\beta}$$

$$- T^\gamma \frac{\partial x^j}{\partial x_\beta}\,\frac{\partial x^\delta}{\partial x^j}\left\{{\alpha \atop \gamma\delta}\right\}$$

$$= \left[T^k\left\{{i \atop kj}\right\} + \frac{\partial T^i}{\partial x^j}\right]\frac{\partial x^\alpha}{\partial x^i}\,\frac{\partial x^j}{\partial x_\beta}$$

$$- T^\gamma \frac{\partial x^\delta}{\partial x_\beta}\left\{{\alpha \atop \gamma\delta}\right\} .$$

Thus $\dfrac{\partial T^\alpha}{\partial x_\beta} + T^\gamma \dfrac{\partial x^\delta}{\partial x_\beta}\left\{{\alpha \atop \gamma\delta}\right\} = \left[\dfrac{\partial T^i}{\partial x^j} + T^k\left\{{i \atop kj}\right\}\right]\dfrac{\partial x^\alpha}{\partial x^i}\,\dfrac{\partial x^j}{\partial x_\beta}$.

Or, using the chain rule and 6)

$$\frac{\partial T^\alpha}{\partial x_\beta} + T^\gamma g^{\beta\delta}\left\{{\alpha \atop \gamma\delta}\right\} = \left[\frac{\partial T^i}{\partial x^j} + T^k\left\{{i \atop kj}\right\}\right]\frac{\partial x^\alpha}{\partial x^i}\,\frac{\partial x^j}{\partial x_\ell}\,\frac{\partial x_\ell}{\partial x_\beta}$$

$$= \left[\frac{\partial T^i}{\partial x^j}\,\frac{\partial x^j}{\partial x_\ell} + T^k\frac{\partial x^j}{\partial x_\ell}\left\{{i \atop kj}\right\}\right]\frac{\partial x^\alpha}{\partial x^i}\,\frac{\partial x_\ell}{\partial x_\beta}$$

$$= \left[\frac{\partial T^i}{\partial x_\ell} + T^k g^{j\ell}\left\{{i \atop kj}\right\}\right]\frac{\partial x^\alpha}{\partial x^i}\,\frac{\partial x_\ell}{\partial x_\beta} .$$

Thus the expression $\left[\frac{\partial T^i}{\partial x_j} + T^k g^{hj} \left\{ {i \atop k\,h} \right\}\right]$ transforms as a tensor of second order, so that 10) is a logical form to be the contravariant derivative.

Notice that 10) can be simplified as follows:

$$\left[\frac{\partial T^i}{\partial x_j} + T^k g^{jh} \left\{ {i \atop k\,h} \right\}\right]$$

$$= \left[\frac{\partial T^i}{\partial x^h} \frac{\partial x^h}{\partial x_j} + T^k g^{jh} \left\{ {i \atop k\,h} \right\}\right]$$

$$= \left[\frac{\partial T^i}{\partial x^h} g^{jh} + T^k g^{jh} \left\{ {i \atop k\,h} \right\}\right]$$

$$= g^{jh}\left[\frac{\partial T^i}{\partial x^h} + T^k \left\{ {i \atop k\,h} \right\}\right]$$

$$= g^{jh} T^i{}_{,h} \,.$$

Thus

11) $$T^{i,j} = g^{jh} T^i{}_{,h} \,.$$

This is consistant with the notation introduced by Ricci and Levi-Civita [2] p. 31.

III. The complexity of the Christoffel symbols is greatly reduced by the requirements:

a) the g_{ij} be symmetric

b) $\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial g_{ik}}{\partial q^j}$.

According to the conventional formulae,

$$[i\ j,\ k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

12)

$$\left\{ {i \atop j\ k} \right\} = g^{ih}\left[\ j\ k,\ h\right].$$

But applying 2) to the above we see that

13) $$[i\ j,\ k] = \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i} .$$

But 2) and 12) imply that

$$\frac{1}{2}\frac{\partial g_{jk}}{\partial q^i} = \frac{1}{2}\frac{\partial g_{ji}}{\partial q^k} = [k\ j,\ i]$$

and combining this with requirement a) the conclusion is reached that the Christoffel symbols of the first kind are <u>triply</u> symmetric; ie:

14) $[i\ j,\ k] = [j\ i,\ k] = [k\ j,\ i] = [k\ i,\ j] = [j\ k,\ i] = [i\ k,\ j]$.

It would follow from their definition that the Christoffel symbols of the second kind are also triply symmetric, but a more important conclusion can be reached about the nature of these symbols. Let us investigate

$$g^{hk}\cdot\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right) = \left\{ {h \atop ij} \right\} .$$

By virtue of 2), a) above, and 6)

$$\left\{ {h \atop ij} \right\} = \frac{1}{2} g^{hk} \frac{\partial g_{jk}}{\partial q^i}$$

$$= \frac{1}{2} g^{hk} \frac{\partial g_{kj}}{\partial x^i}$$

$$= \frac{1}{2} g^{hk} \frac{\partial g_{ij}}{\partial x^k}$$

$$= \frac{1}{2} \frac{\partial x^k}{\partial x_h} \frac{\partial g_{ij}}{\partial x^k}$$

$$= \frac{1}{2} \frac{\partial g_{ij}}{\partial x_h} .$$

Thus we see that for this dual coordinate system the Christoffel symbol of the second kind becomes an exact dual of the symbol of the first kind,

15) $$\left\{ {i \atop jk} \right\} = \frac{1}{2} \frac{\partial g_{jk}}{\partial x_i} .$$

IV. The following result was originally pointed out to the author by Professor Charles C. Torrance.

It is well known that the LaPlacean of a function is given by

16) $$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\; g^{ij} \frac{\partial \phi}{\partial x^j}\right) .$$

To anyone familiar with the tensor notation, it is obvious, since $\nabla^2 \phi$ is an invariant, that some contravariant, second order tensor must be present on the right-hand side. The choice of g^{ij}, to be present inside the parenthesis, is quite logical from a physical consideration of $\nabla^2 \phi$ as div(grad ϕ). However, this expression lacks a symmetry which might be considered desirable.

With the introduction of the covariant coordinate system the expression for the LaPlacean can be written

17) $$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\; \frac{\partial \phi}{\partial x_i}\right)$$

since $g^{ij} \frac{\partial \phi}{\partial x^j} = \frac{\partial \phi}{\partial x_i}$ (see [1] p. 4).

It has come to the author's attention that this idea is quite similar in form to the expression used by some others when discussing the concept of conjugate coordinates in complex analysis.

V. It is quite natural to see what are the consequences of this dualistic coordinate system on the Riemann-Christoffel curvature tensor R_{ijkl}.

18) $$R_{ijkl} = \frac{\partial}{\partial x^l}[ik, j] - \frac{\partial}{\partial x^k}[il, j] + \left\{ {\mu \atop il} \right\} [jk, \mu] - \left\{ {\mu \atop ik} \right\} [jl, \mu] .$$

Applying our results 12) and 13)

$$R_{ijkl} = \frac{\partial}{\partial x^l}[ij, k] - \frac{\partial}{\partial x^k}[ij, l] + \left\{ {\mu \atop il} \right\} [jk, \mu] - \left\{ {\mu \atop ik} \right\} [jl, \mu]$$

$$= \frac{\partial}{\partial x^l}\left(\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right) - \frac{\partial}{\partial x^k}\left(\frac{1}{2}\frac{\partial g_{ij}}{\partial x^l}\right) + \left\{ {\mu \atop il} \right\} [jk, \mu] - \left\{ {\mu \atop ik} \right\} [jl, \mu] .$$

Thus, if the g_{ij} are of class C^2,

19) $$R_{ijkl} = \left\{ {\mu \atop il} \right\} [jk, \mu] - \left\{ {\mu \atop ik} \right\} [jl, \mu] .$$

A series of calculations similar to those in Eisenhart [2] p. 21 shows that the Riemann-Christoffel tensor as given by 19) satisfies the following symmetry equations:

a) $R_{ijkl} = - R_{jikl} = -R_{ijlk}$

20) b) $R_{ijkl} + R_{iljk} + R_{iklj} = 0$

c) $R_{ijkl} = R_{klij}$.

These equations 20) are those which the Riemann-Christoffel tensor normally satisfy (see [2] p. 21). As a consequence there are at most $n^2(n^2 - 1)/12$ independent components of this tensor.

By writing out the 20 distinct components of the tensor for the case $n = 4$, and comparing these terms, it is apparent that there are no further linear relations among the twenty distinct components. Thus, dispite the symmetry of the Christoffel symbols themselves, there is no further symmetry among the components of the Riemann-Christoffel tensor.

It is interesting, and perhaps encouraging, to note that, regarding the Riemann-Christoffel tensor components as a system of differential equations for the g_{ij}, these equations are reduced from second order equations to first order equations in the g_{ij}. However, as will be noted in the next section, the extreme non-linearity of these equations still presents a formidable barrier to their solution.

VI. One interesting result of the preceeding section is obtained by applying the results to a problem spelled out in Bergmann [3]. He takes the Einstein General Field equations in the form

21) $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

and attempts a solution by assuming that each $g_{\mu\nu}$ can be written as a MacLaurin series in a constant λ, and that all but the first two terms can be ignored. As is shown below, our results cancel out the advantages of this method.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$= \left\{ {\rho \atop \nu\rho} \right\}_{,\mu} - \left\{ {\rho \atop \nu\mu} \right\}_{,\rho} - \left\{ {\sigma \atop \nu\mu} \right\}\left\{ {\rho \atop \sigma\rho} \right\} + \left\{ {\sigma \atop \mu\rho} \right\}\left\{ {\rho \atop \nu\sigma} \right\}$$

$$- \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}\Big[\left\{ {\rho \atop \alpha\rho} \right\}_{,\beta} - \left\{ {\rho \atop \alpha\beta} \right\}_{,\rho} - \left\{ {\sigma \atop \alpha\beta} \right\}\left\{ {\rho \atop \sigma\rho} \right\}$$
$$+ \left\{ {\sigma \atop \alpha\rho} \right\}\left\{ {\rho \atop \beta\sigma} \right\}\Big]$$

where the comma indicates partial differentiation.

Bergmann [3] p. 182 assumes

$$g_{\rho\sigma} = \epsilon_{\rho\sigma} + \lambda \underset{1}{h}_{\rho\sigma} + \lambda^2 \underset{2}{h}_{\rho\sigma} + \cdots\cdots$$

where $\epsilon_{\rho\sigma}$ is the metric for Minkowski space and the h's are variables.
He defines $\epsilon^{\rho\sigma} = \epsilon_{\rho\sigma}$; $\underset{1}{h}^{\rho\sigma} = -\epsilon^{\rho\alpha}\epsilon^{\sigma\beta}\underset{1}{h}_{\alpha\beta}$.

Thus $$\left\{ {\rho \atop \nu\rho} \right\} = \frac{1}{2} g^{\rho\tau}\left[g_{\nu\tau,\rho} + g_{\rho\tau,\nu} - g_{\nu\rho,\tau}\right]$$

$$\approx \frac{1}{2}\left(\epsilon^{\rho\tau} + \lambda \underset{1}{h}^{\rho\tau}\right)\left(\lambda \underset{1}{h}_{\nu\tau,\rho} + \lambda \underset{1}{h}_{\rho\tau,\nu} - \lambda \underset{1}{h}_{\nu\rho,\tau}\right).$$

Thus $\left\{ {\rho \atop \nu\rho} \right\}_{,\mu} \approx \frac{1}{2} \lambda\, \epsilon^{\rho\sigma} \left[h_{\sigma\nu,\rho\mu} + h_{\rho\sigma,\nu\mu} - h_{\nu\rho,\sigma\mu} \right]$

and $\left\{ {\rho \atop \mu\nu} \right\}_{,\rho} \approx \frac{1}{2} \lambda\, \epsilon^{\rho\sigma} \left[h_{\sigma\nu,\mu\rho} + h_{\mu\sigma,\nu\rho} - h_{\mu\nu,\sigma\rho} \right]$.

Also $\left\{ {\sigma \atop \rho\sigma} \right\} \left\{ {\rho \atop \mu\nu} \right\} \approx \frac{1}{4} \lambda^2\, \epsilon^{\sigma\tau} \epsilon^{\rho\eta} \left[h_{\rho\tau,\sigma} + h_{\sigma\tau,\rho} - h_{\sigma\rho,\tau} \right] \times$

$$\left[h_{\mu\eta,\nu} + h_{\nu\eta,\mu} - h_{\mu\nu,\eta} \right]$$

and is neglected in comparison to the preceeding terms due to the λ^2.

Thus, in the first approximation, $G_{\mu\nu}$ reduces to

$$G_{\mu\nu} \approx \frac{\lambda}{2} \epsilon^{\rho\sigma} \left[h_{\rho\sigma,\mu\nu} - h_{\nu\rho,\sigma\mu} - h_{\mu\sigma,\nu\rho} + h_{\mu\nu,\rho\sigma} \right]$$
$$- \frac{\lambda}{4} \epsilon_{\mu\nu}\, \epsilon^{\alpha\beta} \epsilon^{\rho\sigma} \left[h_{\rho\sigma,\alpha\beta} - h_{\alpha\rho,\sigma\beta} - h_{\beta\sigma,\alpha\rho} + h_{\alpha\beta,\rho\sigma} \right].$$

But, if covariant coordinates exist, then the terms used to obtain these linearized expressions vanish identically, and

22) $$G_{ij} = \left\{ {s \atop i\,r} \right\} \left\{ {r \atop j\,s} \right\} - \left\{ {s \atop r\,s} \right\} \left\{ {r \atop i\,j} \right\}$$
$$- 1/2\, g_{ij} g^{mn} \left(\left\{ {s \atop m\,r} \right\} \left\{ {r \atop n\,s} \right\} - \left\{ {s \atop r\,s} \right\} \left\{ {r \atop m\,n} \right\} \right).$$

It should be noted that 22) are first order differential equations, though strictly non-linear, whereas the preceeding are second order, linear equations.

Since Bergmann's approximations lead to a known result, when the resultant equations are solved, then if our results are to be compatible, the non-linear terms would have to contribute approximately the same as the linear terms used by Bergmann. I have been unable to confirm or refute this due to the complexity of 22).

VII. Littlewood [4] discusses some connections between tensors and group characters. In particular, he states (p. 128) that the Riemann-Christoffel tensor is of a type whose group character has a Schur function $\{2^2\}$. Because of the orthogonality of the Schur functions this function can be reduced to the sum of three functions,

$$\{2^2\} = [2^2] + [2] + [0]. \tag{23}$$

Since the Riemann-Christoffel tensor can be represented by this function, it too can be reduced to three components, which Littlewood identifies as the conformal curvature tensor, the Ricci tensor G_{ij}, and the scalar curvature.

Since Littlewood gave no proof in [4] that the Riemann-Christoffel tensor is of the type given, the author undertook to construct such a proof and to determine whether the assumption of a covariant coordinate system, and its effect on the form of the Riemann-Christoffel tensor, would alter these results. The major calculations of this proof are given in Appendix A, and the major steps of the analysis are below.

Littlewood considers the group of permutations of n symbols, and from these forms a Frobenius algebra by taking the idempotent linear combinations of these permutations. In Appendix A these idempotent combinations are given explicitly for the case n = 4, and it is demonstrated that each of the elements of the group of permutations can be written explicitly in terms of the elements of the Frobenius algebra.

Since linear combinations of a tensor are still elements of a tensor, it follows that the linear combinations of the Riemann-Christoffel tensor corresponding to the elements of the Frobenius algebra are still components

of a fourth order tensor. A simple such example is the separation of an arbitrary second order tensor b_{ij} into a symmetric and an anti-symmetric tensor

$$b_{ij} = 1/2 \left(b_{ij} + b_{ji}\right) + 1/2 \left(b_{ij} - b_{ji}\right) .$$

Littlewood has shown [6] that by taking a tensor of rank r and the Frobenius algebra of the permutation of r symbols, then corresponding to each irreducible idempotent subalgebra of the algebra, there is a subtensor of the original tensor. The number of independent terms of the subtensor is directly calculatable from the group character of the subalgebra. Corresponding to each permutation of the r symbols there is a Schur function [5], which function is the same for each member of a partion or class. Thus for the permutation of four symbols there are five distinct Schur functions. By use of the orthogonality properties of these functions reduction of the subtensors above can be made.

In appendix A the corresponding subtensors of R_{ijkl} are exhibited. By application of equations 20) it is shown that all of the subtensors are identically zero, except that corresponding to the partition whose Schur function is $\{2^2\}$, which partition has 20 independent components. Since only equations 20) were used in arriving at this reduction, then our form of the Riemann-Christoffel tensor has this same representation and likewise is reducible to the three components according to 23). Thus this property is not altered by the covariant coordinate system.

VIII. Some further investigation which have occurred to the author but which he has not attempted are the following:

(a) In any book on Riemannian geometry several special types of spaces, such as Einstein spaces, recurrent spaces, spaces of constant curvature, and symmetric space, are discussed. (See Wilmore p. 236-239). Is a space which admits a covariant coordinate system one of these special spaces?

(b) With the indicated duality, might it not be profitable to define a symbol such as Γ^{ij}_{k} in some manner analogous to $\Gamma^{k}_{ij} = \left\{ {k \atop i \;\; j} \right\}$?

In particular, this might enable one to give direct meaning to a Riemann-Christoffel tensor R^{ij}_{kl}, from which the Ricci tensor and the scalar curvature are directly obtainable by contraction.

(c) What effect does the triple symmetry of the Christoffel symbols have on the concept of distinct parallelism?

APPENDIX A

Consider the group of permutations on four elements, α, β, γ, δ and write the five possible classes as follows;

$$C_0 = (\alpha)(\beta)(\gamma)(\delta)$$

$$C_1 = (\alpha\beta) + (\gamma\delta) + (\alpha\gamma) + (\beta\delta) + (\alpha\delta) + (\beta\gamma)$$

$$C_2 = (\alpha\beta\gamma) + (\alpha\gamma\delta) + (\alpha\delta\beta) + (\beta\delta\gamma) + [\text{illegible}] + [\text{illegible}] \\ + (\alpha\delta\gamma) + (\beta\gamma\delta)$$

$$C_3 = (\alpha\beta\gamma\delta) + (\alpha\delta\gamma\beta) + (\alpha\beta\delta\gamma) + (\alpha\gamma\delta\beta) + (\alpha\gamma\beta\delta) + (\alpha\delta\beta\gamma)$$

$$C_4 = (\alpha\beta)(\gamma\delta) + (\alpha\gamma)(\beta\delta) + (\alpha\delta)(\beta\gamma).$$

Construct an algebra with five elements as follows;

let x and v be matrices of rank 1, z be a matrix of rank 2, and u and y be matrices of rank 3 whose components are given below.

$$x = \frac{1}{24}\left[C_0 + C_1 + C_2 + C_3 + C_4\right]$$

$$v = \frac{1}{24}\left[C_0 - C_1 + C_2 - C_3 + C_4\right]$$

$$z_{11} = \frac{1}{12}\Big[C_0 + C_4 - (\alpha\beta\gamma) - (\alpha\gamma\delta) - (\alpha\delta\beta) - (\beta\delta\gamma) - (\alpha\beta) - (\gamma\delta) + (\alpha\gamma) + (\beta\delta) \\ - (\alpha\delta\beta\gamma) - (\alpha\gamma\beta\delta) + (\alpha\beta\gamma\delta) + (\alpha\delta\gamma\beta)\Big]$$

$$z_{22} = \frac{1}{12}\Big[C_0 + C_4 - (\alpha\beta\delta) - (\alpha\gamma\beta) - [\text{illegible}] - (\beta\gamma\delta) + [\text{illegible}] + [\text{illegible}] - (\alpha\gamma) - [\text{illegible}] \\ + (\alpha\delta\beta\gamma) + (\alpha\gamma\beta\delta) - (\alpha\beta\gamma\delta) - (\alpha\delta\gamma\beta)\Big]$$

$$z_{12} = \frac{1}{12}\Big[(\alpha\delta) + (\beta\gamma) - (\alpha\gamma) - (\beta\delta) + (\alpha\beta\delta\gamma) + (\alpha\gamma\delta\beta) - [\text{illegible}] - (\alpha\delta\gamma\beta) \\ + (\alpha\beta\gamma) + (\alpha\gamma\delta) + (\alpha\delta\beta) + (\beta\delta\gamma) - [\text{illegible}] - [\text{illegible}] - [\text{illegible}] - [\text{illegible}]\Big]$$

$$z_{21} = \frac{1}{12}\Big[(\alpha\delta) + (\beta\gamma) - (\alpha\beta) - (\gamma\delta) + (\alpha\beta\delta\gamma) + (\alpha\gamma\delta\beta) - [\text{illegible}] - [\text{illegible}] \\ - (\alpha\beta\gamma) - (\alpha\gamma\delta) - (\alpha\delta\beta) - (\beta\delta\gamma) + [\text{illegible}] + [\text{illegible}] + [\text{illegible}] + [\text{illegible}]\Big]$$

$$y_{11} = \frac{1}{8}\Big[C_0 + (\alpha\beta) + (\gamma\delta) - (\alpha\delta\beta\gamma) - (\alpha\gamma\beta\delta) + (\alpha\beta)(\gamma\delta) \\ - (\alpha\gamma)(\beta\delta) - (\alpha\delta)(\beta\gamma)\Big]$$

$$y_{22} = \frac{1}{8}\Big[C_0 + (\alpha\gamma) + (\beta\delta) - (\alpha\beta\gamma\delta) - (\alpha\delta\gamma\beta) + (\alpha\gamma)(\beta\delta)$$
$$- (\alpha\beta)(\gamma\delta) - (\alpha\delta)(\beta\gamma)\Big]$$

$$y_{33} = \frac{1}{8}\Big[C_0 + (\alpha\delta) + (\beta\gamma) - (\alpha\beta\delta\gamma) - (\alpha\gamma\delta\beta) + (\alpha\delta)(\beta\gamma)$$
$$- (\alpha\beta)(\gamma\delta) - (\alpha\gamma)(\beta\delta)\Big]$$

$$y_{21} = \frac{1}{8}\Big[(\alpha\beta\gamma) - (\alpha\gamma\delta) - (\alpha\delta\beta) + (\beta\delta\gamma) - (\alpha\delta) + (\beta\gamma) - (\alpha\gamma\delta\beta) + (\alpha\beta\delta\gamma)\Big]$$

$$y_{23} = \frac{1}{8}\Big[(\alpha\beta\delta) + (\alpha\gamma\beta) - (\alpha\delta\gamma) - (\beta\gamma\delta) + (\alpha\beta) - (\gamma\delta) + (\alpha\gamma\beta\delta) - (\alpha\delta\beta\gamma)\Big]$$

$$y_{31} = \frac{1}{8}\Big[(\alpha\beta\delta) - (\alpha\gamma\beta) - (\alpha\delta\gamma) + (\beta\gamma\delta) - (\alpha\gamma) + (\beta\delta) - (\alpha\delta\gamma\beta) + (\alpha\beta\gamma\delta)\Big]$$

$$y_{32} = \frac{1}{8}\Big[(\alpha\beta\gamma) - (\alpha\gamma\delta) + (\alpha\delta\beta) - (\beta\delta\gamma) + (\alpha\beta) - (\gamma\delta) - (\alpha\gamma\beta\delta) + (\alpha\delta\beta\gamma)\Big]$$

$$y_{12} = \frac{1}{8}\Big[(\alpha\beta\delta) - (\alpha\gamma\beta) + (\alpha\delta\gamma) - (\beta\gamma\delta) + (\alpha\delta) - (\beta\gamma) + (\alpha\beta\delta\gamma) - (\alpha\gamma\delta\beta)\Big]$$

$$y_{13} = \frac{1}{8}\Big[(\alpha\beta\gamma) + (\alpha\gamma\delta) - (\alpha\delta\beta) - (\beta\delta\gamma) + (\alpha\gamma) - (\beta\delta) + (\alpha\beta\gamma\delta) - (\alpha\delta\gamma\beta)\Big]$$

$$u_{11} = \frac{1}{8}\Big[C_0 - (\alpha\beta) - (\gamma\delta) + (\alpha\delta\beta\gamma) + (\alpha\gamma\beta\delta) + (\alpha\beta)(\gamma\delta)$$
$$- (\alpha\delta)(\beta\gamma) - (\alpha\gamma)(\beta\delta)\Big]$$

$$u_{22} = \frac{1}{8}\Big[C_0 - (\alpha\gamma) - (\beta\delta) + (\alpha\beta\gamma\delta) + (\alpha\delta\gamma\beta) + (\alpha\gamma)(\beta\delta)$$
$$- (\alpha\beta)(\gamma\delta) - (\alpha\delta)(\beta\gamma)\Big]$$

$$u_{33} = \frac{1}{8}\Big[C_0 - (\alpha\delta) - (\beta\gamma) + (\alpha\beta\delta\gamma) + (\alpha\gamma\delta\beta) + (\alpha\delta)(\beta\gamma)$$
$$- (\alpha\beta)(\gamma\delta) - (\alpha\gamma)(\beta\delta)\Big]$$

$$u_{12} = \frac{1}{8}\Big[(\alpha\beta\delta) - (\alpha\gamma\beta) + (\alpha\delta\gamma) - (\beta\gamma\delta) - (\alpha\delta) + (\beta\gamma) - (\alpha\beta\delta\gamma) + (\alpha\gamma\delta\beta)\Big]$$

$$u_{13} = \frac{1}{8}\Big[(\alpha\beta\gamma) + (\alpha\gamma\delta) - (\alpha\delta\beta) - (\beta\delta\gamma) - (\alpha\gamma) + (\beta\delta) - (\alpha\beta\gamma\delta) + (\alpha\delta\gamma\beta)\Big]$$

$$u_{21} = \frac{1}{8}\Big[(\alpha\beta\gamma) - (\alpha\gamma\delta) - (\alpha\delta\beta) + (\beta\delta\gamma) + (\alpha\delta) - (\beta\gamma) + (\alpha\gamma\delta\beta) - (\alpha\beta\delta\gamma)\Big]$$

$$u_{23} = \frac{1}{8}\Big[(\alpha\beta\delta) + (\alpha\gamma\beta) - (\alpha\delta\gamma) - (\beta\gamma\delta) - (\alpha\beta) + (\delta\gamma) - (\alpha\gamma\beta\delta) + (\alpha\delta\beta\gamma)\Big]$$

$$u_{31} = \frac{1}{8}\Big[(\alpha\beta\delta) - (\alpha\gamma\beta) - (\alpha\delta\gamma) + (\beta\gamma\delta) + (\alpha\gamma) - (\beta\delta) + (\alpha\delta\gamma\beta) - (\alpha\beta\gamma\delta)\Big]$$

$$u_{32} = \frac{1}{8}\left[(\alpha\beta\gamma) - (\alpha\gamma\delta) + (\alpha\delta\beta) - (\beta\delta\gamma) - (\alpha\beta) + (\gamma\delta) - (\alpha\delta\beta\gamma) + (\alpha\gamma\beta\delta)\right]$$

and define $z^* = z_{11} + z_{22}$, $y^* = y_{11} + y_{22} + y_{33}$, and $u^* = u_{11} + u_{22} + u_{33}$. With these definitions of the elements of the algebra, and defining multiplication of two elements to be successive application of the permutations, it is an easy matter to verify the following products:

$x^2 = x;\quad v^2 = v;\quad z^{*2} = z^*;\quad y^{*2} = y^*;\quad u^{*2} = u^*;$

$xv = vx = 0;$

$xz_{ij} = xy_{ij} = xu_{ij} = z_{ij}x = y_{ij}x = u_{ij}x = 0;$

$vz_{ij} = vy_{ij} = vu_{ij} = z_{ij}v = y_{ij}v = u_{ij}v = 0;$

$z_{11}z_{12} = z_{12}z_{22} = z_{12};\quad z_{22}z_{21} = z_{21}z_{11} = z_{21};$

$y_{11}y_{12} = y_{12}y_{22} = y_{12};\quad y_{22}y_{21} = y_{21}y_{11} = y_{21};$

$y_{13}y_{33} = y_{11}y_{13} = y_{13};\quad y_{31}y_{11} = y_{33}\,y_{31} = y_{31};$

$y_{23}y_{33} = y_{22}y_{23} = y_{23};\quad y_{32}y_{22} = y_{33}y_{32} = y_{32};$

$y_{11}y_{11} = y_{11};\quad y_{22}y_{22} = y_{22};\quad y_{33}y_{33} = y_{33};$

$y_{ij}y_{kl} = 0$ otherwise;

$u_{ij}u_{jj} = u_{ii}u_{ij} = u_{ij};\quad u_{ii}u_{ii} = u_{ii};$

$u_{ij}u_{kl} = 0$ otherwise;

$u_{ij}y_{kl} = 0;\quad y_{ij}u_{kl} = 0;\quad u_{ij}z_{kl} = 0;\ z_{ij}u_{kl} = 0;$

$y_{ij}z_{kl} = z_{ij}y_{kl} = 0.$

Thus these elements form an idempotent algebra. Each of the permutations in the original group may be written in terms of the elements of the algebra as given below. It is noted that for each member of a given class C_i the trace of each of the matrices in the expressions below is the same. The set of traces is called the group character of the class and is set up in the following table.

$$(\alpha\beta)(\gamma\delta) = x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} u + v$$

$$(\alpha\gamma)(\beta\delta) = x + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} u + v$$

$$(\alpha\delta)(\beta\gamma) = x + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u + v$$

$$(\alpha\beta) = x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} y + \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} u - v$$

$$(\gamma\delta) = x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} y + \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} u - v$$

$$(\alpha\gamma) = x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} u - v$$

$$(\beta\delta) = x + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} u - v$$

$$(\alpha\delta) = x + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} u - v$$

$$(\beta\gamma) = x + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} u - v$$

$$(\alpha\beta\gamma) = x + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} u + v$$

$$(\alpha\gamma\delta) = x + \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} u + v$$

$$(\alpha\delta\beta) = x + \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} u + v$$

$$(\beta\delta\gamma) = x + \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} u + v$$

$$(\alpha\beta\delta) = x + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u + v$$

$$(\alpha\gamma\beta) = x + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} u + \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} u + v$$

$$(\alpha\delta\gamma) = x + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} u + \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} u + v$$

$$(\beta\gamma\delta) = x + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} u + v$$

$$(\alpha\beta\gamma\delta) = x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} u - v$$

$$(\alpha\delta\gamma\beta) = x + \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} u - v$$

$$(\alpha\delta\beta\gamma) = x + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} y + \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} u - v$$

$$(\alpha\gamma\beta\delta) = x + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} y + \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} u - v$$

$$(\alpha\beta\delta\gamma) = x + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u - v$$

$$(\alpha\gamma\delta\beta) = x + \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u - v$$

$$(\alpha)(\beta)(\gamma)(\delta) = x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} y + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u + v$$

Class

		C_o	C_1	C_2	C_3	C_4		
algebra element	x	1	1	1	1	1	$\{1^4\}$	Schur function
	y	3	1	0	-1	-1	$\{2\ 1^2\}$	
	z	2	0	-1	0	2	$\{2^2\}$	
	u	3	-1	0	1	-1	$\{3\ 1^2\}$	
	v	1	-1	1	-1	1	$\{4\}$	

The linear combinations of components of the Riemann-Christoffel tensor corresponding to the elements x, y, u, and v can be shown to all be identically zero as follows:

$$x = \frac{1}{24}\left[C_o + C_1 + C_2 + C_3 + C_4\right]$$

$$C_o \sim R_{ijkl}$$

$$C_1 \sim R_{ijkl} + R_{jikl} + R_{kjil} + R_{ilkj} + R_{ljki} + R_{ikjl}$$

$$C_2 \sim R_{kijl} + R_{ljik} + R_{jlki} + R_{iklj} + R_{likj} + R_{jkil} + R_{kjli} + R_{iljk}$$

$$C_3 \sim R_{lijk} + R_{jkli} + R_{kilj} + R_{jlik} + R_{lkij} + R_{klji}$$

$$C_4 \sim R_{jilk} + R_{klij} + R_{lkji}.$$

Applying the identities 20) (a) and (c) we get:

$$C_1 \approx -R_{ijkl} - R_{ijkl} - R_{iljk} - R_{iljk} - R_{iklj} - R_{iklj}$$

$$C_2 \sim R_{iklj} + R_{iklj} + R_{iklj} + R_{iklj} + R_{iljk} + R_{iljk} + R_{iljk} + R_{iljk}$$

$$C_3 \sim -R_{iljk} - R_{iljk} - R_{iklj} - R_{iklj} - R_{ijkl} - R_{ijkl}$$

$$C_4 \sim R_{ijkl} + R_{ijkl} + R_{ijkl} .$$

Thus
$$x = \frac{1}{24}\left[C_o + C_1 + C_2 + C_3 + C_4\right] = 0 .$$

And
$$v = \frac{1}{24}\left[C_o - C_1 + C_2 - C_3 + C_4\right]$$
$$\sim \frac{1}{24}\left[8R_{ijkl} + 8R_{iklj} + 8R_{iljk}\right] .$$

But, applying identity 20) (c) we get

$$v = 0 .$$

And
$$y_{11} = \frac{1}{8}\left[C_o + (\alpha\beta) + (\gamma\delta) - (\alpha\delta\beta\gamma) - (\alpha\gamma\beta\delta) + (\alpha\beta)(\gamma\delta) - (\alpha\gamma)(\beta\delta) - (\alpha\delta)(\beta\gamma)\right]$$
$$\sim \frac{1}{8}\left[R_{ijkl} + R_{jikl} + R_{ijlk} - R_{klji} - R_{lkij} + R_{ijkl}\right.$$
$$\left. - R_{ijkl} - R_{ijkl}\right] = 0 .$$

In a like manner all the elements of y and u can be shown to be zero.

Now consider the components of z.

$$z_{11} = \frac{1}{12}\left[C_o + C_4 - (\alpha\beta\gamma) - (\alpha\gamma\delta) - (\alpha\delta\beta) - (\beta\delta\gamma) - (\alpha\beta) - (\gamma\delta)\right.$$
$$\left. + (\alpha\gamma) + (\beta\delta) - (\alpha\delta\beta\gamma) - (\alpha\gamma\beta\delta) + (\alpha\beta\gamma\delta) + (\alpha\delta\gamma\beta)\right]$$
$$\sim \frac{1}{12}\left[4R_{ijkl} - 4R_{kilj}\ 2R_{ijkl} - 2R_{iljk}\ 2R_{ijkl} - 2R_{iljk}\right]$$
$$= \frac{1}{12}\left[8R_{ijkl} - 4R_{iklj} - 4R_{iljk}\right]$$
$$= R_{ijkl} \quad \text{(by equation 20 b).}$$

Likewise it can be shown that all the other elements of z are non-zero. Since only equations 20) have been used in these results, any fourth order tensor satisfying equations 20) will reduce all components of this Frobenius algebra except z to zero.

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